Momentum, on the other hand, presents us with a more complicated case since we have to deal with a vectorial quantity. The problem is simplified if we treat each of the components of the vector independently. As you recall from the brief presentation in Chapter 3, a component of momentum can be thought of as flowing through matter much like entropy or charge do. The flow of each of the components of momentum results in a flow field like those shown in Chapter 3 and below in Fig. 11.7.

Here are the equations for a single component of momentum. You can build the complete result for all three components by combining the parts. If we choose the x-component, the specific value of x-momentum p_x is the x-component of the velocity. Therefore, we have

$$\mathbf{j}_{px} = \rho v_x \mathbf{v} + \mathbf{j}_{px}{}^{(c)} \tag{11.36}$$

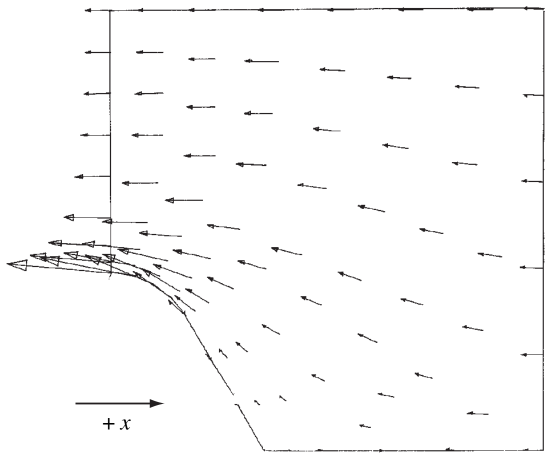

Figure 11.7: Flow pattern of one component of momentum resulting from tension in a flat strip having a notch. The component of momentum whose flow is depicted here is the one identified with the direction of tension. See also Fig. 3.8.

In this case it might be instructive to present all three components of the current density:

$$\begin{aligned} j_{pxx} &= \rho v_x v_x + j_{pxx}{}^{(c)} \\ j_{pxy} &= \rho v_x v_y + j_{pxy}{}^{(c)} \\ j_{pxz} &= \rho v_x v_z + j_{pxz}{}^{(c)} \end{aligned} \tag{11.37}$$

These quantities have a simple graphical representation; j_{pxx}, for example, represents the current density of x-momentum flowing in x-direction, while j_{pxy} is the current density of x-momentum flowing in y-direction (see Fig. 11.7) Since there are three components of current density vectors belonging to the three components of momentum, a total of nine components[7] form the momentum current density tensor.

11.2.5 Transformation of a Surface Integral (Divergence Theorem)

In Section 11.1.2, we transformed a surface integral into an integral over the volume bounded by the surface. We treated the simple example of purely one-dimensional migration of locusts. Since the number of locusts is a scalar quantity, its current density is a vector describing the three possible directions of flow of this fluidlike quantity. If the current density vector has only one component, then locusts move in only one direction. In this case, the locusts flux is

$$I_{Lx} = -\int_{\mathcal{A}} \mathbf{j}_L \cdot \mathbf{n}_x \, dA \overset{def}{=} -\int_{\mathcal{A}} j_{Lx} \, dA$$

The second form on the right has been introduced to shorten the notation. This integral can be transformed into a volume integral according to

$$I_L = -\int_{\mathcal{A}} j_{Lx} \, dA = -\int_{\mathcal{V}} \frac{\partial}{\partial x} j_{Lx} \, dV$$

We used this relation to derive the local form of the equation of balance of locusts above in Section 11.1.2 (see Equ.(11.6)). In this form, the transformation is the simplest example of what is called the *divergence theorem* or *Gauss's theorem*. Let me briefly write down this relation without giving a proof.[8] If we define a current density vector $\mathbf{j}_Q$ on the closed surface of a body, the surface integral can be transformed into an integral over the volume enclosed by the surface:

$$\int_{\mathcal{A}} \mathbf{j}_Q \cdot \mathbf{n} dA = \int_{\mathcal{V}} \nabla \cdot \mathbf{j}_Q dV \tag{11.38}$$

7. This quantity cannot be represented as a vector anymore; rather, it is a tensor which may be written in matrix form

$$\mathcal{J}_p = \begin{pmatrix} \rho v_x v_x + j_{pxx}^{(c)} & \rho v_x v_y + j_{pxy}^{(c)} & \rho v_x v_z + j_{pxz}^{(c)} \\ \rho v_y v_x + j_{pyx}^{(c)} & \rho v_y v_y + j_{pyy}^{(c)} & \rho v_y v_z + j_{pyz}^{(c)} \\ \rho v_z v_x + j_{pzx}^{(c)} & \rho v_z v_y + j_{pzy}^{(c)} & \rho v_z v_z + j_{pzz}^{(c)} \end{pmatrix}$$

The negative conductive part of this quantity is commonly called the *stress tensor*

$$\mathcal{T} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix} = -\begin{pmatrix} j_{pxx}^{(c)} & j_{pxy}^{(c)} & j_{pxz}^{(c)} \\ j_{pyx}^{(c)} & j_{pyy}^{(c)} & j_{pyz}^{(c)} \\ j_{pzx}^{(c)} & j_{pzy}^{(c)} & j_{pzz}^{(c)} \end{pmatrix}$$

while the complete quantity would be called the *momentum current tensor*. The surface integral of a row of the tensor (for one of the components of the coordinate systems) is called the component of the surface force

$$F_x = \int_{\mathcal{A}} \mathbf{T}_x \cdot \mathbf{n} dA$$

($\mathbf{T}_x$ is the first row of the stress tensor), while the surface integral for the stress tensor is the surface force vector

$$\mathbf{F} = \int_{\mathcal{A}} \mathcal{T} \cdot \mathbf{n} dA$$

where $\nabla \cdot \mathbf{j}_Q$ is called the *divergence* of $\mathbf{j}_Q$. In rectangular Cartesian coordinates

$$\nabla \cdot \mathbf{j}_Q = \frac{\partial}{\partial x} j_{Qx} + \frac{\partial}{\partial y} j_{Qy} + \frac{\partial}{\partial z} j_{Qz} \tag{11.39}$$

The divergence of a vector written in component form is often abbreviated as follows:

$$\frac{\partial}{\partial x} j_{Qx} + \frac{\partial}{\partial y} j_{Qy} + \frac{\partial}{\partial z} j_{Qz} \equiv \frac{\partial}{\partial x_i} j_{Qi} \tag{11.40}$$

In this notation it is assumed that a summation is carried out over all indices which appear twice in the same term; x_i, $i = 1,2,3$ stands for the three components (x,y,z) of the coordinate system. In this form, the divergence looks like the expression used in single-dimensional cases. In fact, the simplest examples usually suggest the proper form of more complicated cases.

11.3 THE BALANCE OF MASS

Let us start with the first of the three fluidlike quantities for which we have to obtain laws of balance, namely the amount of substance. The balance of amount of substance is a necessary prerequisite for formulating theories applicable to fluid or otherwise deformable media. If we wish to quantify convective currents associated with processes in open systems, we have to be able to write down the currents of amount of substance. For practical reasons, however, engineers commonly use mass as a substitute for amount of substance, and as long as there are no chemical reactions taking place inside the material, there is no problem in doing so. Therefore, we will use a formulation based on mass.

In the previous section we introduced the concepts and tools needed to formulate the continuum forms of the laws of balance of fluidlike quantities. Starting with the integrated form of the balance of mass

$$\dot{m} = I_m \tag{11.41}$$

Control volume

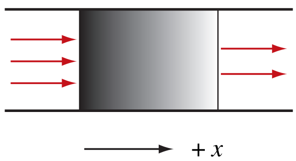

Figure 11.8: Simple one-dimensional flow with respect to an open and stationary control volume. Imagine a fluid flowing in the x-direction only.

we can easily show how to obtain the appropriate local equation applicable to the continuous case. Let us apply this law to a stationary control volume of simple shape (Fig. 11.8) and assume the flow field to be one-dimensional. In this equation, m is the mass inside the control volume, while I_m is the net current of mass across the surface of the control volume. We shall replace the mass by the volume integral of the mass density, and the flux by the surface integral of the flux density, as in Equ.(11.26). With Equ.(11.34) this leads to

$$\frac{d}{dt} \int_{\mathcal{V}} \rho dV + \int_{\mathcal{A}} \rho v dA = 0 \tag{11.42}$$

If we use the divergence theorem for the surface integral and apply the time derivative

8. For a derivation of the divergence theorem see Marsden and Weinstein (1985), Vol. III, p. 927.

to the integrand of the first integral, we obtain

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV + \int_{\mathcal{V}} \frac{\partial}{\partial x}(\rho v) dV = 0$$

or

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \right] dV = 0$$

Since the integral must be zero for arbitrary volumes *V*, the last expression can only be satisfied if the terms in brackets are equal to zero:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad \textbf{(11.43)}$$

You can easily apply the transformations to the more general three-dimensional case

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \qquad \textbf{(11.44)}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \textbf{(11.45)}$$

This looks very similar to the simpler expression. In contrast to Equ.(11.42) which is the integral form of the law of balance of mass, Equ.(11.43) and its counterpart in Equations (11.44) or (11.45) represent the local or differential form of this law. The balance of mass often is called the equation of continuity.

11.4 THE BALANCE OF ENTROPY

Entropy is a scalar quantity just like mass, so the derivation of the local form of the law of balance should lead to a result similar to what we have just seen. Consider as we did in Fig. 11.8, the flow of a fluid in the *x*-direction only. As far as entropy is concerned, we will include conductive and convective transports in the derivation, and production of entropy in irreversible processes. Sources of entropy from radiation, however, will be excluded here. The integral form of the equation of balance of entropy for the control volume in Fig. 11.8 then looks like

$$\dot{S} = I_{S,conv} + I_{S,cond} + \Pi_S \qquad \textbf{(11.46)}$$

If we introduce densities and current densities as in Section 11.2, the law becomes

$$\frac{d}{dt} \int_{\mathcal{V}} (\rho s) dV + \int_{\mathcal{A}} \left(s \rho v + j_S^{(c)} \right) dA = \int_{\mathcal{V}} \pi_S dV \qquad \textbf{(11.47)}$$

s is the *specific entropy* of the fluid, $j_S^{(c)}$ and π_S represent the conductive *entropy current density* and the *density of the entropy production rate*, respectively. Remember that we are dealing with a purely one-dimensional case. If we now apply the transformation of the surface integral, we obtain

$$\int_V \left[\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}\left(s\rho v + j_S{}^{(c)} \right) - \pi_S \right] dV = 0$$

The expression in brackets must be zero, which yields the local form of the law of balance:

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}\left(s\rho v + j_S{}^{(c)} \right) = \pi_S \qquad (11.48)$$

The general three-dimensional case can be written in a form that looks just like the one derived for purely one-dimensional transports. Applying the divergence theorem to the generalized form of Equ.(11.47) yields

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot \left(s\rho \mathbf{v} + \mathbf{j}_S{}^{(c)} \right) = \pi_S \qquad (11.49)$$

or

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x_i}\left(s\rho v_i + j_{Si}{}^{(c)} \right) = \pi_S \qquad (11.50)$$

Extending this result to include the effects of sources from radiation is pretty simple. How this is done will be demonstrated below for the case of momentum (remember that gravity leads to sources of momentum).

11.5 THE BALANCE OF MOMENTUM

Basically, the law of balance of momentum is derived analogously to what you have seen so far. While the fundamental ideas do not change, the current case can be rather complex if we try to deal with it in the most general form. It is therefore all the more important to discuss the simplest possible nontrivial case. Fortunately, purely one-dimensional flow of momentum is meaningful in physical terms, so let us deal with this case in some detail.

One-dimensional convective transport of momentum is a simple concept: if a fluid flows in one direction only, it carries only one single component of momentum. The case of one-dimensional conductive transport is just as well known. Let the direction of fluid flow define the spatial component we are talking about. Having the same component of momentum flowing through the fluid simply means that the material is under compression or tension in the same direction. A frictionless fluid flowing through a straight pipe demonstrates what we mean: the conductive momentum current density of the component parallel to the pipe's axis is the pressure of the fluid.

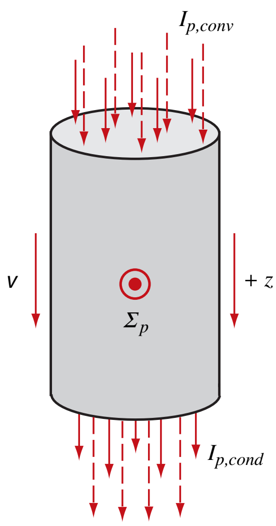

Figure 11.9: Flow lines depicting the convective and conductive transports of momentum are shown together with a source of momentum due to the interaction of the fluid with the gravitational field. The fluid is flowing downward leading to the convective downward flow of momentum together with the fluid (dashed lines). Since the material is under compression, momentum flows conductively in the positive direction (downward; solid lines). With the positive direction as chosen, the gravitational field supplies momentum to the fluid.

In addition to conductive and convective modes of transport, we will consider sources of momentum due to the interaction of the fluid with a field. If you imagine the fluid flowing through a vertical pipe (Fig. 11.9), the action of the gravitational field leads to the flow of momentum of the same (vertical) component directly into or out of the body.

If we collect the different terms, the integral equation of balance of momentum for the z-direction looks like