ON THE USE OF COLLOCATION METHODS FOR THE CONSTRUCTION OF STELLAR MODELS

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Abstract. Collocation with piecewise continuous polynomials is studied for use in the numerical modelling of stellar evolution. Accuracy and convergence of the method are demonstrated for a 5 $M_\odot$ star with a convective core. Collocation should be further studied since it is likely to lead to significant gains in computational efficiency for the construction of stellar models.

1. Introduction

Modern numerical computation of spherically-symmetric stellar evolution is, in almost all cases, carried out by schemes similar to the one developed by L. Henyey and his collaborators (Henyey *et al.*, 1964). This numerical scheme is very powerful, since it allows for the inclusion of detailed opacities, equations of state, and thermonuclear energy generation rates, and also provides for the detailed determination of the spatial and temporal nuclear abundance variations. In addition, the scheme allows the accuracy of a calculation to be increased simply by increasing the number of Lagrangian mesh points in a model. In the collocation method which we describe in this paper features of the Henyey method are preserved; but, in addition, the collocation scheme contains a much greater flexibility which should allow for a significant improvement in computational efficiency. Also, the introduction of a different numerical method will provide a means for the comparison checking of numerical results for complicated phases of stellar evolution.

Collocation methods are now commonly used in aeronautical and chemical engineering, and their use in these areas described in the books by Kopal (1961) and Finlayson (1972). A much more flexible collocation method is described in a paper by Russell and Shampine (1972). Here, they study a collocation scheme based on piecewise continuous polynomials. The scheme described in this paper is patterned after the one of Russell and Shampine.

For our collocation scheme a Lagrangian mesh has been selected. We shall refer to the region between two consecutive mesh points as a *zone*. The unknown solution to the stellar structure equations is, within each zone, written as a polynomial of preselected degree $N+1$. The $N+2$ unknown polynomial coefficients in the solution for each of the dependent variables are determined by requiring that the polynomials exactly satisfy the four stellar structure equations at $N+2$ arbitrarily selected *collocation points* in each zone. Rather than solving for the polynomial coefficients, our

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scheme equivalently involves solving for the unknown variables at the collocation points. The scheme is arranged so that at the first collocation point of the innermost zone and at the last collocation point of the outermost zone of the model, the boundary conditions are introduced. Also, the scheme automatically produces solutions for the radius, luminosity, pressure, and temperature that are continuous at the zone edges, although the first and higher derivatives will have a small mismatch (which is reduced as the numerical parameters are adjusted so as to produce solutions of greater accuracy). At the transition between a convective region and a radiative region, however, a derivative mismatch may be expected in the exact solution.

Since we expect the collocation method to produce accurate solutions, but with fairly large zones, a means must be provided for the accurate determination of the location of the convective-radiative (CR) transitions, which are assumed here to be sharp. We accomplish this by introducing the locations of the CR transitions as additional unknowns and then solving for them simultaneously with the other unknowns. Initially we were concerned about the possibility that this scheme would be slow to converge, or even diverge, but in our tests so far with main sequence-like models with convective cores we have found the CR transition points to converge satisfactorily.

For the systems of differential equations of the form

$$\frac{dy_i}{dx} = F_i[\{y_i\}, x], \qquad (i = 1, \ldots k),$$

collocation is often less efficient than other standard methods if the functions F_i are expressible in short, simple algebraic form. In the stellar evolution problem, however, nearly all of the computational effort goes toward the evaluation of the functions F_i and their derivatives. Hence, it is desirable to use a scheme which allows the number of evaluations of the F_i's and derivatives to be reduced, for a given accuracy. Collocation has the flexibility to accomplish this reduction by variation of the order $N+1$ and the number of zones M. A smaller number of collocation points (in total) is required to achieve a given accuracy as the order $N+1$ is increased. But an increased value of $N+1$ produces a linear system of increased bandwidth. Therefore, it should be possible to minimize the total computational effort at some particular value of $N+1$. This optimum value will depend upon the complexity of the F_i's in relation to the efficiency of the algorithm used for solving the linearized equations.

2. Formulation of the Nonlinear Equations

We wish to obtain solutions of the usual stellar structure equations in spherical symmetry. We proceed as in the Henyey method by solving the structure at a number of discrete times. With respect to time integration, collocation introduces no significant changes in method. Rather, collocation provides a different method for obtaining each stellar model. We omit in this paper a treatment of time-varying nuclear abun-

dances and the term dQ/dt in the heat equation. We shall treat these in the future, but for this paper we wish simply to investigate the convergence and overall accuracy of the collocation scheme as applied to relatively simple stars.

The transformed stellar structure equations are of the form

$$\frac{dy^{(1)}}{dz} = \frac{1}{4\pi}\frac{df}{dz}\left(\frac{dg}{dy^{(1)}}\right)^{-1} g^{-2}\varrho^{-1}, \tag{1}$$

$$\frac{dy^{(2)}}{dz} = \frac{df}{dz}\left(\frac{dk}{dy^{(2)}}\right)^{-1} \varepsilon, \tag{2}$$

$$\frac{dy^{(3)}}{dz} = -\frac{G}{4\pi}\frac{df}{dz}\left(\frac{dh}{dy^{(3)}}\right)^{-1} fg^{-4}, \tag{3}$$

$$\frac{dy^{(4)}}{dz} = -\frac{G}{4\pi}\frac{df}{dz}\left(\frac{dl}{dy^{(4)}}\right)^{-1} fl\nabla g^{-4}h^{-1}; \tag{4}$$

with the boundary conditions

$$C_1(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}) = 0, \tag{5}$$

$$C_2(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}) = 0, \tag{6}$$

at the inner boundary of the model (usually near the center); and

$$B_1(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}) = 0, \tag{7}$$

$$B_2(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}) = 0, \tag{8}$$

at the outer model boundary (the base of the stellar envelope, usually). The dependent variables $y^{(1)}, \ldots, y^{(4)}$ are related to the radius r, local luminosity L, total pressure P, and temperature T by

$$r = g(y^{(1)}), \qquad L = k(y^{(2)}), \qquad P = h(y^{(3)}), \qquad T = l(y^{(4)}), \tag{9}$$

where the exponential-like functions g, k, h, and l are introduced in order to reduce the range of variation of the unknowns. The independent variable z is related to the Lagrangian mass coordinate m by

$$m = f(z).$$

The quantities ϱ and ε are the density and thermonuclear energy generation rate, respectively and are known functions of the local values of P, T and composition. ∇ is the local value of the logarithmic temperature gradient

$$\nabla \equiv \frac{d\ln T}{d\ln P},$$

and, in general, is a function of the local values of the unknowns and of the local composition. G is the constant of universal gravitation.

The range of z between the inner and outer boundaries of the model is divided into M zones by the set of points $a_1, a_2, \ldots, a_{M+1}$. These points should be selected so as to

equalize among all of the zones the maximum change of any of the four dependent variables across the zones. This is easy to automate in an evolutionary sequence of models since it is sufficient to do this zoning on the basis of the last computed model. The a_i's are regarded as fixed parameters, except for those few which are designated as CR transition points. These CR points will be modified during the iteration process. Within each zone a variable ξ is defined ($0 \leqslant \xi \leqslant 1$) such that in the p-th zone z is related to ξ by

$$z = a_p + \xi(a_{p+1} - a_p). \tag{10}$$

In each zone ξ is now regarded as the independent variable.

Within each zone the unknown functions $y^{(1)}, \ldots, y^{(4)}$ are each written as a polynomial of degree $N+1$

$$y^{(r)}(\xi) = \sum_{m=0}^{N+1} d_m^{(r)} \xi^m, \qquad (r = 1, \ldots, 4), \tag{11}$$

where we have suppressed indication of the zone number p. We select $N+2$ collocation points ξ_j in each of the zones, where $\xi_0 = 0$ and $\xi_{N+1} = 1$. At these points the unknown functions have the values

$$\begin{aligned} y_i^{(r)} &= \sum_{m=0}^{N+1} d_m^{(r)} \xi_i^m \\ &= \sum_{m=0}^{N+1} B_{im}\, d_m^{(r)}, \qquad (r = 1, \ldots, 4, \quad i = 0, \ldots, N+1). \end{aligned} \tag{12}$$

Now we replace the derivatives $\mathrm{d}y^{(r)}/\mathrm{d}\xi_i$ occurring in the differential equations by linear combinations of the $y_i^{(r)}$'s in the zone. Differentiating Equation (11) at the collocation points we have

$$\begin{aligned} \left.\frac{\mathrm{d}y^{(r)}}{\mathrm{d}\xi}\right|_{\xi=\xi_i} &= \sum_{m=0}^{N+1} m \xi_i^{m-1} d_m^{(r)} \\ &= \sum_{m=0}^{N+1} C_{im} d_m^{(r)}, \qquad (r = 1, \ldots, 4, \quad i = 0, \ldots, N+1). \end{aligned} \tag{13}$$

Equations (12) and (13) define the matrices B and C. Eliminating the quantities $d_m^{(r)}$ between Equations (12) and (13) we find that

$$\left.\frac{\mathrm{d}y^{(r)}}{\mathrm{d}\xi}\right|_{\xi=\xi_i} = \sum_{j=0}^{N+1} A_{ij} y_j^{(r)}, \qquad (r = 1, \ldots, 4, \quad i = 0, \ldots, N+1), \tag{14}$$

where the matrix A is calculated from

$$\mathsf{A} = \mathsf{C} \cdot (\mathsf{B})^{-1}. \tag{15}$$

By use of Equation (14) the differential equations at the collocation points become the following set of nonlinear algebraic equations:

$$\sum_{m=0}^{N+1} A_{im} y_m^{(1)} - \left[\frac{1}{4\pi}\frac{df}{dz}\left(\frac{dg}{dy^{(1)}}\right)^{-1} g^{-2}\varrho^{-1}\right]_{z=z_i} (a_{p+1} - a_p) = 0,$$
$$(p = 1, \ldots, M, \quad i = 1, \ldots, N+1), \quad (16)$$

$$\sum_{m=0}^{N+1} A_{im} y_m^{(2)} - \left[\frac{df}{dz}\left(\frac{dk}{dy^{(2)}}\right)^{-1} \varepsilon\right]_{z=z_i} (a_{p+1} - a_p) = 0,$$
$$(p = 1, \ldots, M, \quad i = 1, \ldots, N+1), \quad (17)$$

$$\sum_{m=0}^{N+1} A_{im} y_m^{(3)} + \left[\frac{G}{4\pi}\frac{df}{dz}\left(\frac{dh}{dy^{(3)}}\right)^{-1} g^{-4} f\right]_{z=z_i} (a_{p+1} - a_p) = 0,$$
$$(p = 1, \ldots, M, \quad i = 0, \ldots, N), \quad (18)$$

$$\sum_{m=0}^{N+1} A_{im} y_m^{(4)} + \left[\frac{G}{4\pi}\frac{df}{dz}\left(\frac{dl}{dy^{(4)}}\right)^{-1} fl\,\nabla\, g^{-4} h^{-1}\right]_{z=z_i} (a_{p+1} - a_p) = 0,$$
$$(p = 1, \ldots, M, \quad i = 0, \ldots, N), \quad (19)$$

where

$$z_i = a_p + \xi_i(a_{p+1} - a_p)$$

within the p-th zone. Notice that Equations (16) and (17) are written for the *last* $N+1$ collocation points in the p-th zone while Equations (18) and (19) are written for the *first* $N+1$ collocation points in the zone. The first ($i=0$) and the last ($i=N+1$) collocation points of a zone are 'shared' with the neighboring zones. The unknowns at these zone edges belong to the polynomials expansions of two adjacent zones. Hence continuity is automatically incorporated into the solutions. The boundary conditions, Equations (5) and (6), are added to the set of equations for the first zone (at $i=0$) and the outer boundary conditions, Equations (7) and (8) are added to the set of equations at the Mth zone (at $i=N+1$).

At a convective-radiative transition the location of the zone edge a_p' is unknown. Hence, an additional equation

$$\nabla_{\text{rad}}(y^{(1)}, y^{(2)}, a_p', y^{(3)}, y^{(4)}) - \nabla_{\text{crit}}(y^{(1)}, y^{(2)}, a_p', y^{(3)}, y^{(4)}) = 0 \quad (20)$$

must be added to the set of equations. ∇_{rad} is the usual radiative temperature gradient and ∇_{crit} is the critical gradient for the onset of convective instability.

For order $N+1$ collocation with M zones and K CR transitions we have $4[M(N+1)+1]+K \equiv N_T$ equations and N_T unknowns. The set of equations to be solved is given by Equations (5)–(8), and (15)–(20). The number of collocation points involved is $N_C \equiv M(N+1)+1$. After the unknowns at the collocation points are determined, the solution elsewhere is easily evaluated using Equation (11) and

$$d_j^{(r)} = \sum_{m=0}^{N+1} B_{jk}^{-1} y_k^{(r)} \quad (21)$$

in each of the M zones.

3. Solution of Nonlinear Equations

The unknowns are arranged in a one-dimensional array in the following sequence: *zone 1:* $y_0^{(1)}, \ldots, y_0^{(4)}, y_1^{(1)}, \ldots, y_{N+1}^{(1)}, y_{N+1}^{(2)}$; *zone 2:* $y_0^{(3)}, y_0^{(4)}, \ldots, y_{N+1}^{(1)}, y_{N+1}^{(2)}$; ...; *zone q:* $y_0^{(3)}, y_0^{(4)}, \ldots, y_{N+1}^{(1)}, y_{N+1}^{(2)}$; *CR transition:* a_q; *zone q+1:* $y_0^{(3)}, y_0^{(4)}, \ldots, y_{N+1}^{(1)}, y_{N+1}^{(2)}$; ...; *zone M:* $y_0^{(3)}, y_0^{(4)}, \ldots, y_{N+1}^{(1)}, \ldots, y_{N+1}^{(4)}$. The nonlinear equations are also arranged in the corresponding order. The number of unknown CR transition points will depend on the evolutionary state and type of star being modelled.

The nonlinear equations can be solved by Newton's method. The Jacobian of the system is illustrated schematically in Figure 1. The nonzero elements are located in sparse blocks along the main diagonal, which, for interior zones, have size $4(N+2)$ by $4(N+1)$. Block dimensions are different for blocks adjacent to either boundary or a CR transition point (or both). We have found it convenient to incorporate Equations

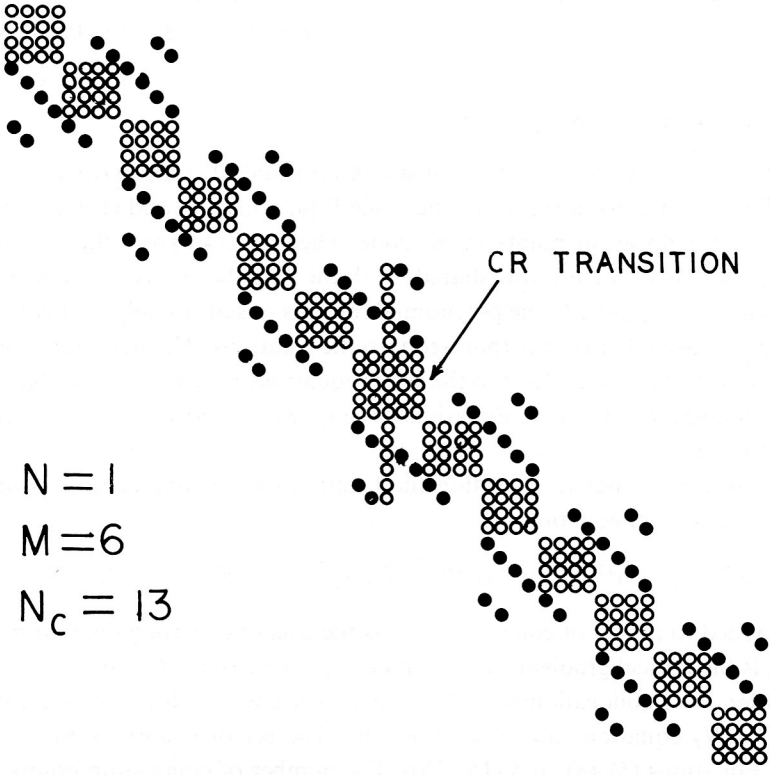

Fig. 1. Diagram showing the location of generally nonzero matrix elements of the Jacobian matrix. Open circles indicate matrix elements which change from iteration to iteration. They represent derivatives of the left hand sides of Equations (16)–(20) with respect to the dependent variables $y_i^{(1)}$, $y_i^{(2)}$, $y_i^{(3)}$, $y_i^{(4)}$, a_p', where the subscript i designates the collocation point at which a given equation (represented by a row in the diagram) is written. Filled circles indicate matrix elements (elements of the matrix A defined by Equation (15)) which are constant from iteration to iteration. They are derivatives of the polynomial derivative of Equations (16)–(19) with respect to the several $y_m^{(1)}, y_m^{(2)}, y_m^{(3)}, y_m^{(4)}, m \neq i$.

(5)–(8) and (20) into adjacent blocks. This produces blocks of eight different types altogether. We have solved the linearized equations with a gaussian block elimination scheme similar to the one described by Henyey *et al.* (1964). This is probably not the most efficient scheme for this system, although it is simple to program since the matrix elements can be calculated one block at a time as the solution proceeds, with no need to store the entire matrix in the computer. We have used a subroutine for general linear systems for solving each block. This can probably be improved upon in the future.

Special care must be taken in evaluating derivatives with respect to a_q', a CR transition point. Equation (10) shows that for a fixed value of ξ, as a_q' changes, so does the physical location of the collocation points in the adjacent zones. These changes must be incorporated into the corresponding matrix elements. Otherwise the iteration process may not converge.

4. Results from Test Calculations

Our initial tests of collocation were made on a 5 $M_\odot$ Main Sequence star of uniform composition. The equation of state used was that of an ideal gas with radiation. Equilibrium CN cycle hydrogen burning was included with $Z_{CN}=0.01$, and Cox and Stewart (1970) opacities were used with $X=0.7$, $Z=0.01$. This choice of material properties was not intended to generate models which are as physically accurate as possible, but only to produce models close to Main Sequence stars. Convection was assumed to occur and dominate energy transfer if $\nabla_{rad} \geqslant \nabla_{ad}$, where ∇_{ad} is the usual adiabatic temperature gradient.

Our test calculations did not include model envelopes or atmospheres. Rather, the collocation method was applied throughout the entire star. This was done in order to save the trouble of programming envelope routines at this time. But it also provided a test of the method over a large range of the unknowns. The boundary conditions used at the surface were

$$L_{surf} = 4\pi r_{surf}^2 \sigma T_{surf}^4, \tag{22}$$

$$P_{surf} = \frac{8}{3}\frac{\sigma}{c} T_{surf}^4, \tag{23}$$

where σ is the usual radiation constant. The boundary condition given by Equation (23) is rather arbitrary. It merely insures a low photospheric pressure. The innermost edge of zone 1 was located at mass fraction 2×10^{-6}. The central conditions $r_c=0$ and $L_c=0$ were put into the form of Equations (7) and (8) by use of the following equations at the first collocation point:

$$1 - \frac{3m_1}{4\pi r_1^3 \varrho_c} = 0, \tag{24}$$

$$1 - \frac{\varepsilon_c m_1}{L_1} = 0, \tag{25}$$

where the subscript 'c' refers to the center and the subscript '1' refers to the first collocation point of the mesh. ϱ_c and ε_c are known functions of the central pressure and temperature which are in turn functions of P_1 and T_1 as determined by truncated Taylor expansions at the first collocation point. Alternatively we could have introduced the center as an additional mesh point and retained the first two stellar structure Equations, (16) and (17), in the first zone.

For the calculations described here we have used the following variables:

$$y^{(1)} = \sinh^{-1}(r/1.0 \times 10^9 \text{ cm}),$$

$$y^{(2)} = \sinh^{-1}(L/4.0 \times 10^{29} \text{ erg s}^{-1}),$$

$$y^{(3)} = \exp(T/1.0 \text{ K}),$$

$$y^{(4)} = \exp(P/1.0 \text{ dyne cm}^{-2}),$$

$$z = \ln\left(1 - \frac{m}{M}\frac{1}{(1+\eta)}\right),$$

where m/M is the mass fraction and the 'surface spreading' parameter η is 10^{-8}.

For a standard of comparison we first calculated a model using the fitting method and a very accurate subroutine for step-by-step integration of differential equations. This comparison model has the following properties:

$$L_{\text{surf}} = 2.41 \times 10^{37} \text{ erg s}^{-1},$$

$$r_{\text{surf}} = 1.75 \times 10^{11} \text{ cm},$$

$$T_c = 2.68 \times 10^8 \text{ K},$$

$$P_c = 7.12 \times 10^{17} \text{ dyne cm}^{-2},$$

$$z_{\text{core}} = -0.25654,$$

where z_{core} gives the position of the edge of the convective core. We believe that this comparison model is an accurate solution of the equations to 1 part in 10^4, or better, everywhere.

To assess the accuracy of the collocation method we converged a number of models with different numbers of zones (placed so that the maximum change of a dependent variable across a zone was equalized among all of the zones), with different orders $N+1$, and with three different choices for the placement of the collocation points within the zones. These three choices were equally spaced points (E), the zeros of the shifted Chebyshev polynomial of degree N (C), and the zeros of the shifted Legendre polynomial of degree N (L), in addition to the points $\xi=0$ and $\xi=1$. Many other choices for the placement of the collocation points are possible. DeBoor and Swartz (1973) and Russell (1974) have shown that the use of Legendre polynomial zeros is more accurate than the use of equally spaced points. Chebyshev zeros are expected to be the optimum choice for high order $N+1$ (see, for example Lanczos, 1957). The accuracy of the converged models is shown in Table I. In this table N_C is the total number of collocation points in the model and M is the number of zones. The last five

TABLE I

N_C	N	M	Points	$\Delta y_{\text{surf}}^{(1)}$	$\Delta y_{\text{surf}}^{(2)}$	$\Delta y_c^{(3)}$	$\Delta y_c^{(4)}$	Δz_{core}
91	1	45	E	+0.0308	+0.0459	+0.0060	−0.0186	+0.0034
94	2	31	E	0.0203	0.0234	0.0052	+0.120	0.0023
93	3	23	E	0.0064	0.0084	0.0077	0.0080	0.0018
91	4	18	E	+0.0209	+0.0246	+0.0098	+0.0141	+0.0028
94	2	31	C	−0.0185	−0.0218	+0.0059	−0.0060	−0.0032
93	3	23	C	−0.0049	−0.0052	−0.0025	−0.0039	−0.00080
91	4	18	C	−0.0069	−0.0054	−0.0030	−0.0045	−0.00044
94	2	31	L	−0.0018	−0.0013	+0.0055	+0.0020	−0.00047
93	3	23	L	−0.0001	−0.0002	+0.0016	+0.0004	−0.00006
91	4	18	L	−0.0014	−0.0006	+0.0024	+0.0006	−0.00004
201	1	100	E	+0.0082	+0.0138	−0.0024	+0.0041	+0.0011
205	2	68	E	0.0062	0.0081	0.0018	0.0024	0.0010
205	3	51	E	0.0016	0.0023	0.0001	0.0011	0.00042
201	4	40	E	+0.0017	+0.0017	+0.0002	+0.0010	+0.00035
205	2	68	C	−0.0045	−0.0066	+0.0024	−0.0015	−0.00066
205	3	51	C	−0.0007	−0.0010	+0.0000	−0.0004	+0.00002
201	4	40	C	+0.0000	−0.0003	−0.0001	−0.0001	+0.00017
205	2	68	L	+0.0001	−0.0002	+0.0003	+0.0002	+0.00011
205	3	51	L	0.0001	+0.0001	+0.0000	+0.0000	+0.00016
201	4	40	L	0.0001	+0.0000	+0.0001	+0.0001	+0.00017

columns of the table give the errors in the solution at the surface and center of the model and in the location of the convective core edge. This error is defined by, for example,

$$\Delta y_{\text{surf}}^{(1)} = (y_{\text{surf}}^{(1)})_{\text{collocation}} - (y_{\text{surf}}^{(1)})_{\text{exact}},$$

where the 'exact' value is obtained from the comparison model.

An examination of Table I reveals the following general trends. (1) Increasing the number of zones, for a given order, increases the accuracy. It should be noted here that if separate stellar envelopes were calculated in order to provide the outer boundary condition, then about two-thirds as many collocation points would provide the same accuracy as our models since many of the zones were introduced in order to represent the rapid drop in pressure near the surface. (2) An increase in order $N+1$ generally brings an increase in accuracy, for a given total number of collocation points. The 4th line of Table I is a puzzling exception; we have not been able to find the reason for this loss of accuracy. (3) The use of Chebyshev or Legendre points produces a significant increase in accuracy as compared to evenly spaced points and this increase in accuracy comes at *no* cost in computational effort. For small values of N the Legendre points seem to give more accurate solutions than do Chebyshev points.

The amount of computer time t required to compute one iteration is given approximately by

$$t \approx t_1\{M(N+1)+1\} + \\ + t_2 M\{64(N+1)^3 + 48(N+1)^2 + 36(N+1)\},$$

where the first term gives the amount of time required for the evaluation of the matrix elements of the Jacobian and the second term is the amount of time required for the solution of the linearized system. The numbers t_1 and t_2 will depend on the efficiency of the program, the complexity of the 'physics' used, and the sophistication of the algorithm for solving the linearized system. In the program written for the test calculations described in this paper the ratio t_1/t_2 was about 700.

Table II shows the convergence of the most accurate of the models represented in Table I. This convergence behavior is typical of the convective-core models we have examined. The second column of the table gives average correction for all of the unknowns for each iteration. The next four columns show the maximum correction for each variable type, and the last column shows the corrections for the convective zone edge. We find convergence difficulties only when the number of collocation points is small and the order is large, e.g., $N_C=25$, $N+1=5$.

TABLE II

Iteration	$\lvert\delta y\rvert_{av}$	$\lvert\delta y^{(1)}\rvert_{max}$	$\lvert\delta y^{(2)}\rvert_{max}$	$\lvert\delta y^{(3)}\rvert_{max}$	$\lvert\delta y^{(4)}\rvert_{max}$	δz_{core}
1	1.2×10^{-2}	6.8×10^{-2}	1.9×10^{-1}	2.1×10^{-1}	5.7×10^{-1}	-2.3×10^{-2}
2	3.8×10^{-3}	4.1×10^{-3}	5.5×10^{-3}	2.6×10^{-2}	6.2×10^{-3}	-2.1×10^{-3}
3	3.0×10^{-4}	8.8×10^{-5}	1.4×10^{-4}	1.5×10^{-4}	2.9×10^{-5}	-1.8×10^{-5}
4	2.0×10^{-8}	1.6×10^{-6}	5.6×10^{-8}	5.5×10^{-9}	2.2×10^{-8}	-1.3×10^{-9}
5	5.3×10^{-12}	4.3×10^{-10}	1.6×10^{-11}	1.8×10^{-12}	6.8×10^{-13}	-1.6×10^{-14}

We conclude this section with some programming notes. We have found that an accurate solution of the linearized system, with iterative improvement where necessary, can actually save computer time by reducing the total number of iterations necessary to solve the nonlinear equations. Secondly, the locations of the CR transitions must be monitored by the program during the iteration process so that re-zoning of adjacent zones can be carried out if the 'current' CR transition 'passes through' a neighboring (fixed) zone edge. This re-zoning can be done accurately and easily since Equations (11) and (21) provide the necessary interpolation formulae.

5. Conclusion

We find the collocation method described here to be clearly suitable for the calculation of detailed stellar evolution. This method should be further studied since it has sufficient adjustability to allow significant improvements in computational efficiency in two areas. The first is the location of the collocation points. There is an enormous flexibility

in their placement, and an automatic proceedure for determining their placement is clearly needed. Secondly, zoning efficiency can be improved. Here we have zoned all four dependent variables identically. This is inefficient for variables which do not change much across a zone. Independent zoning for each of the dependent variables should be possible since collocation provides the necessary interpolation scheme. Independent zoning would, however, involve more complicated linearized systems.

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